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ON SOME APPLICATIONS OF BESSEL'S FUNCTIONS WITH PURE IMAGINARY INDEX.

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§ 1. Bessel's functions, although occurring in investigations in almost every branch of mathematical physics, find in a certain potential problem a more varied and thorough application than they do in any other single problem of equal simplicity with which I am acquainted. In a great variety of problems (diffraction, electrical induction, etc.) only the functions of the zeroth and first orders $J_0(x)$ and $J_1(x)$ occur. In others (solution of Kepler's equation) only Bessel's functions $J_n(x)$ with entire index n are used. Even in a large class of problems concerning small vibrations of elastic bodies, to which fuller reference will be made hereafter, these functions still appear in a comparatively restricted form.

The potential problem alluded to above is the following:

Given a solid S bounded by two coaxial cylinders of revolution, two planes through the axis of these cylinders, and two planes perpendicular to this axis, it is required to find a function $V(x, y, z)$ which 1) *everywhere within S satisfies Laplace's equation $\Delta V = 0$, and is finite, continuous, and single valued, together with its first space derivatives, and* 2) *assumes on the surface of S arbitrarily assigned values.**

* In order not to complicate matters, I leave out of consideration the more general boundary condition in which $aV + b\dfrac{\partial V}{\partial n}$ assumes arbitrarily assigned values on the surface of S (a and b being constants, and n indicating the direction of the normal to the surface at any point). Moreover, although the special case where $a = 0$ could be solved by our method with as much ease as the case we consider where $b = 0$, the solution of the general case could not in general be carried to completion. The problem of the flow of heat in the solid S will naturally suggest itself to the reader as including the potential problem stated above as a special case; namely, that in which the flow of heat has become permanent. This general problem of the non-stationary flow of heat will, however, at least in the simple case where each point of the surface is maintained at a constant temperature, involve no essentially new applications of Bessel's functions.

It is the method of treating this problem by *development in series* with which we are concerned.

The problem here stated may be regarded as a special case of a similar one in which it is required to find a potential within a solid bounded not like S by two cylindrical and four plane faces, but by six confocal quadric surfaces, or more generally still, by six confocal cyclides. This last mentioned general problem was for the first time enunciated and solved by Professor F. Klein, in a course of lectures on Lamé's functions held at the university of Göttingen in the winter of 1889–90. It was there pointed out that most, if not all, of the simpler potential problems which have already been solved by the method of development in series are at most but slightly modified special cases of this general problem (cf. a note on this subject inserted in the Göttinger Nachrichten, March, 1890).

For an exposition of this theory I must refer to a work of my own,* where, however, much that concerns special cases had to be greatly compressed owing to want of space. While I hope before long to be able to give from the same point of view a much more extended and thorough exposition of the whole theory, whereby the relations between various well known problems will stand out in clear light, as it would otherwise be impossible to make them; it still seems worth while to present, in a form accessible to those who are not familiar with the general theory just mentioned, an interesting special case which has not as yet been completely treated.

I have, in the following pages, frequently been obliged to depart from the form which Professor Klein is accustomed to give to his theory; but the points of view from which I work are his. That the present article is incomplete in several points will be readily seen. This I have not tried to conceal, but rather to emphasize, in order that the difficulties which remain may be the sooner cleared away.

We will now proceed to the special potential problem stated above. Our first step is to break it up into six simpler potential problems, whose solutions, when added together, give the required complete solution. Each of these partial problems is identical with the original problem, except that the values of the potential are given, as above, arbitrarily on one face of the solid only; while on the other five faces the potential is required to assume the value zero. One of the six faces of S, and of course each time a different one, thus stands out in each of the six partial problems as being connected with a more complicated boundary condition than the other five. I will speak of this face as the *exceptional* face.

* Ueber die Reihenentwickelungen der Potentialtheorie. Am 4. Juni, 1891, von der philosophischen Facultät der Universität Göttingen gekrönte Preisschrift. Göttingen, 1891.

These six partial problems arrange themselves naturally in three groups of two problems each. In the first of these groups the exceptional face is one of the two curved faces of S. In the second it is one of the two parallel plane faces, or, as I will call them, *ends* of S. In the third it is one of the two remaining plane faces,—*meridian planes.*

Corresponding to these three groups of problems we shall have three different types of solution, while the two problems of any one group have solutions of the same form, which differ in the constants etc. involved.

The first steps towards the solution of any of these six partial problems are the same; and although they are well known, I will run through them briefly.

a) Cylinder coordinates z, r, φ are introduced in place of Cartesian, so that z denotes the distance from an initial plane parallel to the ends of S, r the perpendicular distance from the axis of the cylindrical faces of S, and φ the angle which a meridian plane, i. e. a plane through this axis, makes with an initial meridian plane. The faces of S may then be denoted by the simple equations

$$r = r_1,\quad r = r_2,\quad z = z_1,\quad z = z_2,\quad \varphi = \varphi_1,\quad \varphi = \varphi_2\,;$$

or still more simply, after a proper choice of our system of cylinder coordinates, by

$$r = r_1,\quad r = r_2,\quad z = 0,\quad z = z_0,\quad \varphi = 0,\quad \varphi = \varphi_0.$$

We will assume, in what follows, that the cylinder coordinates have been thus chosen.

Transformed to this system of coordinates Laplace's equation takes on the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0\,.$$

b) Our next step is to look for *Lamé's products** corresponding to this system of cylinder coordinates; *i. e.* solutions of the potential equation of the form

$$V = Z\,.\,R\,.\,\Phi\,,$$

* I use this name, with Prof. Klein, not to denote that the three factors of the product are Lamé's functions, or special cases thereof, although this is of course true in the present case; but rather as a general name for products of this sort, whatever be the nature of their factors.

where Z, R, Φ are functions respectively of z, r, φ only. It is readily found that this will be a potential when, and only when, the three factors are solutions, respectively, of the following differential equations, in which k and n represent any two constants:

$$\frac{d^2 Z}{dz^2} = k^2 Z,$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[k^2 - \frac{n^2}{r^2}\right] R = 0,$$

$$\frac{d^2 \Phi}{d\varphi^2} = -n^2 \Phi.$$

The general solutions of these equations are

$$Z = A_1 e^{kz} + B_1 e^{-kz},$$

$$R = A_2 J_n(kr) + B_2 J_{-n}(kr),$$

$$\Phi = A_3 \sin n\varphi + B_3 \cos n\varphi;$$

except that when n is zero or a real integer Bessel's functions of the second kind must be introduced in place of $J_{-n}(kr)$.

From this point on each of our six problems will require a separate treatment. I will here confine myself to the consideration of three of these problems, chosen one from each of the three above mentioned groups, as the other three problems would offer no new points of interest. I will choose the following three problems:

1) That in which $z = z_0$ is the exceptional face;
2) " " " $r = r_2$ " " " ";
3) " " " $\varphi = \varphi_0$ " " " ".

We will now consider these three problems in succession.

§ 2. Our first step in problem 1) is to pick out those Lamé's products which satisfy all of our boundary conditions except the one on the exceptional face $z = z_0$. In order that our Lamé's product should vanish when $\varphi = 0$, we must have $B_3 = 0$. In order that it should vanish when $z = 0$, A_1 and B_1 must be equal with opposite signes; so that

$$Z = A_1 (e^{kz} - e^{-kz}) = 2A_1 \sinh (kz).$$

In order that the product should vanish when $\varphi = \varphi_0$ we must let $n = \nu\,\pi/\varphi_0$ where ν is an integer, which we may consider positive. In order

that it should vanish when $r = r_1$ we must let

$$\frac{B_2}{A_2} = - J_{\frac{\nu\pi}{\phi_0}}(kr_1) / J_{-\frac{\nu\pi}{\phi_0}}(kr_1).$$

Finally, in order that it should vanish when $r = r_2$, we must take k as one of the roots of the transcendental equation

$$J_{\frac{\nu\pi}{\phi_0}}(kr_1) \,.\, J_{-\frac{\nu\pi}{\phi_0}}(kr_2) - J_{\frac{\nu\pi}{\phi_0}}(kr_2) \,.\, J_{-\frac{\nu\pi}{\phi_0}}(kr_1) = 0 \,.$$

It can easily be shown* that this equation has an infinite number of real roots. Arranging these in order of magnitude, we will denote them by $k_{\nu,1}$, $k_{\nu,2}$, $k_{\nu,3}, \ldots$†

From the Lamé's products which we have thus determined we can now build the following series :

$$\sum_1^\infty{}^{\nu} \sum_1^\infty{}^{\mu} C_{\mu,\nu} \sinh(k_{\nu,\mu}z) \,.\, \sin\left[\frac{\nu\pi}{\varphi_0}\varphi\right] \,.\, \Big[J_{-\frac{\nu\pi}{\phi_0}}(k_{\nu,\mu}r_1) \,.\, J_{\frac{\nu\pi}{\phi_0}}(k_{\nu,\mu}r) - J_{\frac{\nu\pi}{\phi_0}}(k_{\nu,\mu}r_1) \,.\, J_{\frac{\nu\pi}{\phi_0}}(k_{\nu,\mu}r)\Big] \,,$$

which will vanish when $r = r_1$, $r = r_2$, $\varphi = 0$, $\varphi = \varphi_0$, $z = 0$, and which we will assume satisfies Laplace's equation, as it certainly would if it consisted of only a finite number of terms. We must still determine the coefficients $C_{\mu,\nu}$, which are the only undetermined constants now left, in such a way that the above series shall reduce, when $z = z_0$, to a development of the arbitrarily assigned function of φ and r which expresses the value the potential is to have on the exceptional face $z = z_0$.

When this is done the above series, assuming that it converges, is the solution of our problem 1).

The method of thus determining the coefficients $C_{\mu,\nu}$ is well known and need not be discussed here; but it may be noted that it depends upon the relation

$$\int_{r_1}^{r_2} r \cdot E_{\frac{\nu\pi}{\phi_0}}(k_{\nu,\mu}r) \cdot E_{\frac{\nu\pi}{\phi_0}}(k_{\nu,\mu'}r)\, dr = 0 \,,$$

* The proposition stated above is an easy generalization of the more familiar proposition concerning the roots of the equation $J_\nu(kr) = 0$. For a proof of this last named proposition cf. Riemann-Hattendorf, Partielle Differentialgleichungen, p. 267.

† We confine our attention, as we obviously have a right to do, to *positive* roots. It is evident that $-k_{\nu,1}, -k_{\nu,2}, \ldots$ will also be roots of the equation.

where $k_{\nu,\mu}$ and $k_{\nu,\mu'}$ are any two *different* roots of the equation $E_\nu(kr_2) = 0$, and where for brevity we have written,

$$E_{\frac{\nu\pi}{\phi_0}}(kr) = J_{-\frac{\nu\pi}{\phi_0}}(kr_1) \cdot J_{\frac{\nu\pi}{\phi_0}}(kr) - J_{\frac{\nu\pi}{\phi_0}}(kr_1) \cdot J_{-\frac{\nu\pi}{\phi_0}}(kr).$$

We now see that *the Bessel's functions involved in the problems of type* 1) *have real argument x and real index n.*

There is a simple problem in the vibration of membranes which bears the closest resemblance to the type of potential problem just discussed. This problem concerns the vibration of a plane homogeneous isotropic membrane uniformly stretched and acted upon by no external forces, when the fixed boundary of the membrane consists of two arcs of concentric circles and two radii of the circles. This problem will, like type 1) of our potential problem, introduce Bessel's functions with argument and index both real.

§ 3. We come now to our potential problem 2), in which $r = r_2$ is the exceptional face. Here, again, our first step is to find all the Lamé's products which vanish on all the faces of S except on the exceptional face. In order that the Lamé's product should vanish when $r = 0$, $\varphi = 0$, $\varphi = \varphi_0$, we must, as before, take

$$A_1 = -B_1, \quad B_3 = 0, \quad n = \frac{\nu\pi}{\varphi_0},$$

where ν is an integer. In order that it should vanish when $z = z_0$ we must further have $k = \dfrac{i\kappa\pi}{z_0}$, where κ is an integer. Finally, in order that the product should vanish when $r = r_1$ we must let,

$$\frac{A_2}{B_2} = -J_{-\frac{\nu\pi}{\phi_0}}\left[\frac{i\kappa\pi}{z_0}r_1\right] \Big/ J_{\frac{\nu\pi}{\phi_0}}\left[\frac{i\kappa\pi}{z_0}r_1\right].$$

From the Lamé's products thus determined we now form the following series:

$$\sum_1^\infty{}_\nu \sum_1^\infty{}_\kappa C_{\nu,\kappa} \cdot \left[J_{-\frac{\nu\pi}{\phi_0}}\left[\frac{i\kappa\pi}{z_0}r_1\right] \cdot J_{\frac{\nu\pi}{\phi_0}}\left[\frac{i\kappa\pi}{z_0}r\right] - J_{\frac{\nu\pi}{\phi_0}}\left[\frac{i\kappa\pi}{z_0}r_1\right] \cdot J_{-\frac{\nu\pi}{\phi_0}}\left[\frac{i\kappa\pi}{z_0}r\right]\right] \cdot \sin\frac{\nu\pi\varphi}{\varphi_0} \cdot \sin\frac{\kappa\pi z}{z_0}.$$

This will then be the potential we are seeking, if the coefficients $C_{\nu,\kappa}$ are so determined that when $r = r_2$ the series reduces to a development of the

arbitrary function of φ and z which expresses the value the potential is to take on on the exceptional face $r = r_2$. This development is, however, merely a Fourier's series of two arguments, whose coefficients are determined by means of well known formulæ.

In this type of potential problem we have to do with Bessel's functions whose indices are real but whose arguments are pure imaginary.

Now we shall find that *in our potential problem* 3) *we are concerned with Bessel's functions whose index and argument are both pure imaginary.* As these functions are nowhere explicitly considered, as far as I know, I will consider them briefly before taking up the special application of them we have in hand.

§ 4. The Bessel's function $J_n(x)$ may be written in the form,

$$J_n(x) = \frac{x^n}{2^n \cdot \Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \ldots\right].$$

When the argument x here becomes pure imaginary $J_n(x)$ will still be real except for the factor i^n, which can be removed without causing our function to cease being a solution of Bessel's equation. When, however, the index n becomes pure imaginary, x being either real or pure imaginary, $J_n(x)$ will become imaginary, and cannot now be made real by the removal of a factor not depending upon x. Originally $J_n(x)$ and $J_{-n}(x)$ were chosen for use as being the simplest of the particular solutions of Bessel's equation. Now, however, it will be advantageous to choose two other *real* particular solutions.

For this purpose I will first drop the factor $\frac{1}{2^n\,\Gamma(n+1)}$ which appears in $J_n(x)$* and write the particular solution of Bessel's equation thus obtained $\{J_n(x)\}$. Of course a second solution of the equation linearly independent of this one will be $\{J_{-n}(x)\}$. When n has a pure imaginary value, x being still supposed real, both of these solutions will, as has already been said, have complex values. We may, however, form from them the following particular solutions, which are real:

$$H_n(x) = \frac{1}{2}\left[\{J_n(x)\} + \{J_{-n}(x)\}\right], \quad I_n(x) = \frac{1}{2i}\left[\{J_n(x)\} - \{J_{-n}(x)\}\right].$$

* There seems at first sight to be but little advantage to be gained by retaining this factor as the *practical* advantage depends upon the *relationes inter contiguas* (see my paper on Bessel's functions of the second kind in the January number of this journal, especially p. 89), and these would now connect our functions with others having a *complex* index. On the other hand for *theoretical* considerations the functions $J_n(x)$ and $J_{-n}(x)$ although having complex values would probably still in most cases be the most convenient. This question, however, deserves a more careful consideration than we can give it here.

These functions may easily be expressed in real form; but in order that the law according to which their developments proceed may be clear, it is advisable first to introduce the following abbreviation:

Let p and q be any positive integers such that $q \leqq p$. *We will use the symbol $(p)_q$ to denote the sum of all the different products which can be formed by multiplying together q of the p factors $p, p-1, p-2, \ldots, 3, 2, 1$.*

The simplest cases will be

$$(p)_p = p\,!,$$

$$(p)_{p-1} = p\,!\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p}\right],$$

$$(p)_1 = 1 + 2 + 3 + \ldots + p.$$

We will further give, as definitions, $(p)_0 = 1$ and $(p)_q = 0$ when $q > p$ or when $q < 0$. We shall now obviously have the relation,

$$(p)_q = (p-1)_q + p\,(p-1)_{q-1},$$

by means of which the following table may be easily calculated:

Table of the values of $(p)_q$.

$q =$	$p = 1$	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1	1	3	6	10	15	21	28	36
2		2	11	35	85	175	322	546
3			6	50	225	735	1960	4536
4				24	274	1624	6769	22449
5					120	1764	13132	67284
6						720	13068	118124
7							5040	109584
8								40320

For a further account of these numbers the reader is referred to *Schlömilch's* "Compendium der hoheren Analysis," Vol. II, Chap. I, from which the above table is taken.

Writing now $n = i\nu$, it will readily be found, that

$$H_{i\nu}(x) = \cos(\nu \log x) \,.\, S_1(x) + \sin(\nu \log x) \,.\, S_2(x),$$

and

$$I_{i\nu}(x) = -\cos(\nu \log x) \,.\, S_2(x) + \sin(\nu \log x) \,.\, S_1(x);$$

where $S_1(x)$ and $S_2(x)$ denote the following power series:*

$$S_1(x) = 1 - \frac{1}{4(1^2+\nu^2)} x^2 + \frac{(2)_2 - (2)_0\,\nu^2}{4^2 .\, 2!\,(1^2+\nu^2)(2^2+\nu^2)} x^4$$

$$- \frac{(3)_3 - (3)_1\,\nu^2}{4^3 .\, 3!\,(1^2+\nu^2)(2^2+\nu^2)(3^2+\nu^2)} x^6 + \frac{(4)_4 - (4)_2\,\nu^2 + (4)_0\,\nu^4}{4^4 .\, 4!\,(1^2+\nu^2)\ldots(4^2+\nu^2)} x^8$$

$$- \frac{(5)_5 - (5)_3\,\nu^2 + (5)_1\,\nu^4}{4^5 .\, 5!\,(1^2+\nu^2)\ldots(5^2+\nu^2)} x^{10} + \frac{(6)_6 - (6)_4\,\nu^2 + (6)_2\,\nu^4 - (6)_0\,\nu^6}{4^6 .\, 6!\,(1^2+\nu^2)\ldots(6^2+\nu^2)} x^{12} - \ldots,$$

$$S_2(x) = -\frac{\nu}{4 .\,(1^2+\nu^2)} x^2 + \frac{(2)_1\,\nu}{4^2 .\, 2!\,(1^2+\nu^2)(2^2+\nu^2)} x^4$$

$$- \frac{(3)_2\,\nu - (3)_0\,\nu^3}{4^3 .\, 3!\,(1^2+\nu^2)\ldots(3^2+\nu^2)} x^6 + \frac{(4)_3\,\nu - (4)_1\,\nu^3}{4^4 .\, 4!\,(1^2+\nu^2)\ldots(4^2+\nu^2)} x^8$$

$$- \frac{(5)_4\,\nu - (5)_2\,\nu^3 + (5)_0\,\nu^5}{4^5 .\, 5!\,(1^2+\nu^2)\ldots(5^2+\nu^2)} x^{10} + \frac{(6)_5\,\nu - (6)_3\,\nu^3 + (6)_1\,\nu^5}{4^6 .\, 6!\,(1^2+\nu^2)\ldots(6^2+\nu^2)} x^{12} - \ldots.$$

When not only the index $n = i\nu$, but also the argument $x = iz$, is pure imaginary, we find as two real linearly independent Bessel's functions,†

$$\overline{H}_{i\nu}(iz) = \cos(\nu \log z) \,.\, S_1(iz) + \sin(\nu \log z) \,.\, S_2(iz),$$

$$\overline{I}_{i\nu}(iz) = -\cos(\nu \log z) \,.\, S_2(iz) + \sin(\nu \log z) \,.\, S_1(iz).$$

We may notice that H and $\overline{H}$ are even, I and $\overline{I}$ odd, functions of their indices.

§ 5. We will now return to our potential problem 3) and we will write the factor R of the Lamé's product in the form

$$R = A_2\, \overline{H}_n(kr) + B_2\, \overline{I}_n(kr).$$

* I will leave the question of the convergency of the series S_1 and S_2 quite open. The series for the functions H and I will, however, certainly be convergent if we agree to break off with the same power of x in S_1 as in S_2.

† I use the term Bessel's function here and elsewhere to denote *any* solution of Bessel's equation, not merely the two normal solutions $J_n(x)$ and $J_{-n}(x)$ as is usually done.

In order now that our Lamé's products should vanish when $z = 0$, $z = z_0$, $\varphi = 0$, and $r = r_1$ we must let

$$A_1 = -B_1, \quad k = \frac{i\varkappa\pi}{z_0}, \quad B_3 = 0\,, \quad \frac{A_2}{B_2} = -\bar{I}_n\left[\frac{i\varkappa\pi}{z_0}r_1\right] \Big/ \bar{H}_n\left[\frac{i\varkappa\pi}{z_0}r_1\right].$$

In order finally that the Lamé's products should vanish when $r = r_2$ we must take n as a root of the transcendental equation,

$$\bar{I}_n\left[\frac{i\varkappa\pi}{z_0}r_1\right] \cdot \bar{H}_n\left[\frac{i\varkappa\pi}{z_0}r_2\right] - \bar{H}_n\left[\frac{i\varkappa\pi}{z_0}r_1\right] \cdot \bar{I}_n\left[\frac{i\varkappa\pi}{z_0}r_2\right] = 0\,.$$

It will be shown in the next section that this equation has an infinite number of pure imaginary roots, and no others. These roots, arranged in order of magnitude, we will denote by $i\nu_{\kappa,1}$, $i\nu_{\kappa,2}$, $i\nu_{\kappa,3}$, ...

We have thus picked out an infinite number of Lamé's products, and from them we will form the series

$$\sum_1^\infty{}^{\kappa} \sum_1^\infty{}^{\mu}\, C_{\kappa,\mu} \,.\, \sinh\left(\nu_{\kappa,\mu} \,.\, \varphi\right) \,.\, \sin\left[\frac{\varkappa\pi}{z_0}z\right] \cdot \Big[\bar{I}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r_1\right] \cdot \bar{H}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r\right]$$
$$- \bar{H}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r_1\right] \cdot \bar{I}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r\right]\Big].$$

In order that this should be the required potential, we have now only to determine the coefficients $C_{\kappa,\mu}$ in such a way that when $\varphi = \varphi_0$ the above series shall reduce to a development of the arbitrary function $f(r, z)$ which expresses the value the potential is to take on on the exceptional face of S.

We have, then, the problem of developing the function $f(r, z)$ in the form

$$f(r, z) = \sum_1^\infty{}^{\kappa} \sum_1^\infty{}^{\mu}\, A_{\kappa,\mu} \,.\, \sin\left[\frac{\varkappa\pi}{z_0}z\right] \cdot E_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r\right],$$

where, for brevity, has been put

$$E_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r\right] = \bar{I}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r_1\right] \cdot \bar{H}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r\right] - \bar{H}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r_1\right] \cdot \bar{I}_{i\nu_{\kappa,\mu}}\left[\frac{i\varkappa\pi}{z_0}r\right].$$

This function E might also be defined, without the aid of a formula, as that Bessel's function, with index $i\nu_{\kappa,\mu}$ and argument $\frac{i\varkappa\pi}{z_0}r$, which vanishes when $r = r_1$; a definition which, to be sure, still leaves us free to multiply our E by any factor not involving r.

In order to determine the coefficients $A_{\kappa,\mu}$, we may write the above development in the form

$$f(r, z) = \sum_1^\infty{}^{\kappa} \left[\sum_1^\infty{}^{\mu} A_{\kappa,\mu} \,.\, E_{i\nu_{\kappa,\mu}} \left[\frac{ix\pi}{z_0} r \right] \right] . \sin \left[\frac{x\pi}{z_0} z \right].$$

If now, for a moment, we regard r as constant, we see that this is merely a Fourier's development of a function of a single variable; and that, accordingly, its coefficients will be given by the well known formula; so that we shall have

$$\sum_1^\infty A_{\kappa,\mu} \,.\, E_{i\nu_{\kappa,\mu}} \left[\frac{ix\pi}{z_0} r \right] = \frac{2}{z_0} \int_0^{z_0} f(r, z) \,.\, \sin \left[\frac{x\pi}{z_0} z \right] . \, dz.$$

The second member of this equation is a perfectly definite function of the argument r only, which we will denote by $F(r)$; and the constants $A_{\kappa,\mu}$ must now be determined as the coefficients in the development of $F(r)$ according to the Bessel's functions E. It should be noticed that in this series the successive Bessel's functions have the same argument but different indices; while in the ordinary development of an arbitrary function according to Bessel's functions (cp. p. 141) the index remains the same throughout the series, while the argument varies. We require accordingly a new formula here for the determination of the coefficients. In order to find this formula we make use of the following proposition, whose proof will be given in the next section:—

If $E_\nu(ix)$ is a Bessel's function which vanishes when $x = x_1$ and if $i\nu_1$, $i\nu_2$, $i\nu_3$, . . . are the roots of the equation $E_\nu(ix_2) = 0$, then,

$$\int_{x_1}^{x_2} \frac{1}{x} E_{i\nu_p}(ix) \,.\, E_{i\nu_q}(ix) \,.\, dx = 0,$$

except when $\nu_p = \pm\, \nu_q$.

By means of this proposition we get, at once, for the coefficients in the above expansion, the formula

$$A_{\kappa,\mu} = \frac{\displaystyle\int_{r_1}^{r_2} \frac{1}{r} F(r) \,.\, E_{i\nu_{\kappa,\mu}} \left[\frac{ix\pi}{z_0} r \right] . \, dr}{\displaystyle\int_{r_1}^{r_2} \frac{1}{r} \left[E_{i\nu_{\kappa,\mu}} \left[\frac{ix\pi}{z_0} r \right] \right]^2 dr.}.$$

We have now solved our potential problem 3),* as the coefficients $C_{\kappa,\mu}$ are

* It still remains to prove that this solution is correct, by proving first that the series we here obtain converge, and secondly that the series we obtain for our potential really satisfies Laplace's equation.

given by the formula

$$A_{\kappa,\mu} = C_{\kappa,\mu} \,.\, \sinh\,(\nu_{\kappa,\mu} \,.\, \varphi_0).$$

§ 6. We still have two propositions to discuss, whose truth was assumed in the last section.

First, the integral formula stated near the close of the section.

The equation which is satisfied by Bessel's functions with imaginary index $i\nu$ and imaginary argument ix, is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 - \nu^2)\, y = 0$$

By the introduction of the new variable $t = \log x$ this equation assumes the simpler form

$$\frac{d^2y}{dt^2} = (x^2 - \nu^2)\, y\,,$$

a form which, as we shall see, deserves a more important place than it receives in most treatises on Bessel's functions.*

We get, then, for the functions $E_{i\nu_p}(ix)$ and $E_{i\nu_q}(ix)$ the equations

$$\frac{d^2E_{i\nu_p}(ix)}{dt^2} = (x^2 - \nu_p^2) \,.\, E_{i\nu_p}(ix)\,,$$

$$\frac{d^2E_{i\nu_q}(ix)}{dt^2} = (x^2 - \nu_q^2) \,.\, E_{i\nu_q}(ix).$$

Multiply these equations respectively by $E_{i\nu_q}(ix)$ and $E_{i\nu_p}(ix)$, and subtract, obtaining

$$E_{i\nu_q}(ix) \,.\, \frac{d^2E_{i\nu_p}(ix)}{dt^2} - E_{i\nu_p}(ix) \,.\, \frac{d^2E_{i\nu_q}(ix)}{dt^2} = (\nu_q^2 - \nu_p^2)\, E_{i\nu_p}(ix) \,.\, E_{i\nu_q}(ix)\,.$$

Multiplying this equation through by dt, and integrating between the limits $x = x_1$ and $x = x_2$, gives us

$$\left[E_{i\nu_q}(ix) \,.\, \frac{dE_{i\nu_p}(ix)}{dt} - E_{i\nu_p}(ix) \,.\, \frac{dE_{i\nu_q}(ix)}{dt}\right]_{x=x_1}^{x=x_2}$$

$$= (\nu_q^2 - \nu_p^2) \int_{x=x_1}^{x=x_2} E_{i\nu_p}(ix) \,.\, E_{i\nu_q}(ix) \,.\, dt\,.$$

* For, of course, there is a similar form when index and argument are real; see Riemann-Hattendorf, loc. cit.

Now since $E_{i\nu_q}(ix)$ and $E_{i\nu_p}(ix)$ vanish at both limits x_1 and x_2 the whole left hand member of this equation is zero. If, then, we introduce x again in place of t, we get

$$(\nu_q^2 - \nu_p^2) \int_{x_1}^{x_2} x^{-1} E_{i\nu_p}(ix) \,.\, E_{i\nu_q}(ix) \,.\, dx = 0 \,,$$

which proves the proposition, since by hypothesis ν_q^2 and ν_p^2 are not equal.*

By an obvious extension of the method just employed we may also find the formula

$$\int_{x_1}^{x_2} x^{-1} \left(E_{i\nu_p}(ix)\right)^2 . \, dx = -\frac{x_2}{2\nu_p}\left[\frac{dE_{i\nu}(ix)}{d\nu} \,.\, \frac{dE_{i\nu}(ix)}{dx}\right]_{\substack{x = x \\ \nu = \nu_p^2}}$$

which may be used, if it should seem desirable, in the expression for the coefficients $C_{\kappa,\mu}$ which was found in the last section.

Finally, we must show that the transcendental equation which occurred in the last section (p. 146) really has an infinite number of pure imaginary roots, as was there stated. This follows at once from the following proposition:—

If $E_{i\nu}(ix)$ is a Bessel's function which vanishes when $x = x_1$, then the equation in ν

$$E_{i\nu}(ix_2) = 0 \,,$$

where x_2 is a real constant, has an infinite number of real roots, but no imaginary roots.

To prove this proposition, let us first consider the curve

$$y = E_{i\nu}(ix) \,,$$

in which we regard ν for the moment as a real constant. The nature of this curve will most easily be seen, if at first we take not x but $t = \log x$ as abscissa. Making use, now, of the form

$$\frac{d^2y}{dt^2} = (x^2 - \nu^2)\, y$$

of Bessel's equation, we see at once, by considerations similar to those em-

* Attention should be called to the striking similarity between the proposition just proved and formula 47 *f.*, Heine's Handbuch der Kugelfunctionen, Vol. I, p. 255. The two propositions are, however, essentially different, as will readily be seen; and, moreover, Heine's is much more special, inasmuch as his limits of integration are not x_1 and x_2, but 0 and ∞.

ployed by Riemann-Hattendorf,* that the curve whose abscissa is t and ordinate y will consist, to the left of the point $t = \log \nu$, of an infinite number of arches lying alternately above and below the axis of t, much like the curve $y = \sin t$, except that the length and height of all the arches are not the same. To the right of the point $t = \log \nu$ the curve will cross the axis of t either only once or not at all, having, to speak roughly, the character of a hyperbolic sine or cosine curve. Having now pictured to ourselves this curve, it is easy to return to the curve of which x is the ordinate. This curve we need consider only for positive values of x, and for these values we can describe it as follows:

The curve $y = E_{i\nu}(ix)$ has in the neighborhood of the point $x = 0$ an infinite number of infinitely short arches. These arches lie alternately above and below the axis of x, and become longer as we recede from the origin. The curve retains this oscillatory character until $x = \nu$, and then either recedes indefinitely from the axis of x without meeting it again, or first completes the arch which was begun before $x = \nu$, and then after crossing the axis of x recedes indefinitely from it.†

We thus see that *the equation in x, $E_{i\nu}(ix) = 0$, has an infinite number of roots between zero and ν, and at most one root greater than ν.*

This proposition, however, although important and interesting, is not the one we wanted; for we have supposed ν constant and x variable, whereas we wished to let ν vary and to give to x the constant value x_2. We have, however,

* It may be of assistance to consider Bessel's equation as written above as the equation of motion of a particle moving in a straight line under the action of a centre of force situated upon this line, the force being directly proportional to the distance y, but also varying with the time, and even changing from an attractive to a repulsive force at the time $t = \log \nu$.

† Another possibility is the case lying between the two just mentioned, and which corresponds to the curve $y = e^{-t}$ just as these corresponded to the curves $y = \cosh t$ and $y = \sinh t$; namely, that in which the arch which began before $x = \nu$ extends out to $x = \infty$; so that the curve has the axis of x (not, as will readily be seen, a line parallel to this axis) as an asymptote. Moreover, by leaving the index ν unchanged, and merely taking different solutions of our Bessel's equation, we can get each of these three types of curves, as will be at once obvious if we consider that, in the mechanical problem suggested in the foot note above, the different solutions of Bessel's equation correspond to different initial positions and velocities of the particle.

We thus get the important proposition,

Whatever the value of ν, Bessel's equation always has one solution which vanishes when $x = +\infty$.

The point $x = \infty$ is an "irregular" point of Bessel's equation, i. e. a point about which the solutions of the equation cannot be developed in power series. The determination of the behavior of the solutions in the neighborhood of these points is therefore difficult (unless we have them expressed as semi-convergent series), hence the importance of the result just found, which may without difficulty be generalized as follows:

If we approach an irregular point of any (degenerate) Lamé's equation from one side along the axis of reals, either; (a) ALL *the solutions of the equation will remain finite, having an infinite number of oscillations in the neighborhood of the point; or (b)* ONE *solution will vanish at the point in question while all solutions linearly independent of it become infinite.*

This proposition was omitted for want of space from my essay quoted on page 138.

no convenient direct method of defining E as a function of ν. It will, therefore, be best to proceed as follows: Consider both ν and x as variable, and construct the surface $y = E_{i\nu}(ix)$, in which y is the vertical, x and ν the horizontal coordinates. This we can do by means of the plane sections $\nu =$ const., each one of which gives one of the curves described above. We have now only to consider how this curve changes as we let ν increase from 0, to obtain the following description of the surface:—

The line $y = 0$, $x = x_1$ lies in the surface $y = E_{i\nu}(ix)$, dividing it into two parts of somewhat different character. Each part, however, consists of an infinite number of ridges separated by valleys, all of which, as ν increases, run up asymptotically to the line $y = 0$, $x = x_1$.* (See Fig. I, which represents qualitatively the trace of this surface on the plane $y = 0$.)

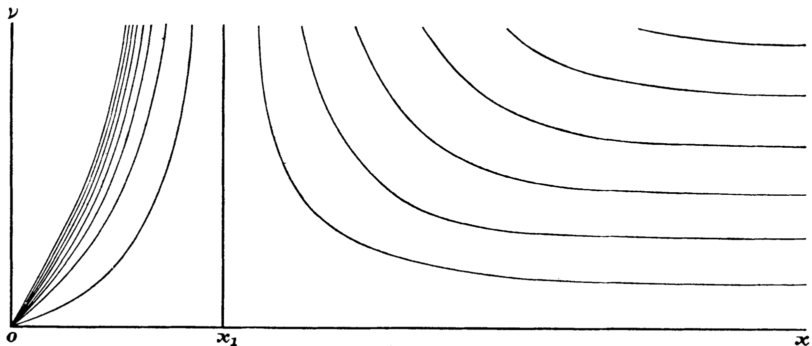

Fig. I.

It is evident that any plane section $x = x_2$ will cut through an infinite number of these ridges, and will therefore be an oscillating curve, consisting of an infinite number of arches lying alternately above and below the line $y = 0$; and hereby our proposition is proved that the equation $E_{i\nu}(ix_2) = 0$ has an infinite number of real roots.

But we have also proved more than this, as will be seen by a slight consideration of our surface; namely, that *if $\nu_1, \nu_2, \nu_3, \ldots$ are the roots arranged in order of magnitude (beginning with the smallest), the Bessel's function $E_{i\nu_\mu}(ix)$ vanishes exactly $\mu - 1$ times in the interval between $x = x_1$ and $x = x_2$, the ends of this interval being excluded.*

This is a case of the *theorem of oscillation* originally due to Sturm (Liou-

* To the left of this line these ridges and valleys all start from the origin. To the right they start from points whose ν coordinates are finite, whose x coordinates, however, are infinite; so that these ridges and valleys have asymptotes, all different, parallel to the axis of x.

ville's Journal, Vol. 1), and greatly extended and emphasized by Prof. Klein (Mathematische Annalen, Vol. 18).*

To complete the proof by showing that our equation has no imaginary roots, we first notice that it is impossible that it should have any *pure* imaginary roots, for we should then have a Bessel's function with real index and pure imaginary argument, which, as can readily be proved,† cannot meet the axis of x more than once. Accordingly, since by definition it meets the axis of x when $x = x_1$, it cannot do so also when $x = x_2$.

We can now easily prove, by a well known method due to Poisson, that no complex roots can exist. For if ν were such a complex root its conjugate ν would also be a root.‡ We know then from the integral formula proved at the beginning of this section, that

$$\int_{x_1}^{x_2} x^{-1} \, . \, E_{i\nu}(ix) \, . \, E_{i\bar{\nu}}(ix) \, . \, dx = 0$$

(since $\nu \pm \nu$ is not zero). But $E_{i\nu}(ix)$ and $E_{i\nu}(ix)$ have for every real value of x between x_1 and x_2 conjugate imaginary values,§ so that their product is *positive.* Hence every element of the definite integral is positive, and the definite integral cannot vanish, as we know it must; and thus our supposition that there is a complex root leads to an absurdity.

§ 7. Let us return to the three types of potential problem which we have now in theory completely solved. Although a solid of the generality of our solid S is seldom or never mentioned in works on the theory of the potential, still problems in no essential respect less general than our types 1) and 2) are well known to physicists. This fact will be explained by the following considerations:—

If in our solid S we let the angle φ_0, which the two meridian faces make with each other, increase until it reaches the value 2π, our solid will assume

* The geometrical proof which I have here given of this theorem seems better suited for conveying a real understanding of the proposition than such analytic proofs as, following Sturm, are usually given. In a similar way we can prove other interesting properties of Bessel's functions. For instance, *Between two consecutive roots* ν_μ *and* $\nu_{\mu+1}$ *of the equation just discussed, lies one, and only one, root of the equation* $\frac{dE_{i\nu}(ix_2)}{d\nu} = 0$.

† This is obvious if we refer to our mechanical problem of page 150 for we now have *repulsion* under the action of which our particle cannot pass through the centre of force more than once.

‡ This follows from the "principle of symmetry" employed by Prof. Schwarz in Crelle Vol. 70 with such effect. In its simplest form this principle tells us that *if in an analytic function, to a continuous succession of real values of the argument there correspond real values of the function, then to any pair of conjugate imaginary values of the argument will correspond a pair of conjugate imaginary values of the function.*

§ This also follows directly from the above mentioned principle.

the form of a ring bounded by two coaxial cylinders and two parallel planes perpendicular to the axis of these cylinders. This ring shaped solid is, however, cut open at one point by a plane which passes through the axis of the cylinders, and which forms the two meridian faces of our solid. If, now, we modify our potential problem by simply removing these meridian faces, i. e. by requiring that as we cross over these faces our potential V should be single valued, and continuous together with its first space derivatives, we shall have arrived at the form of problem usually considered by physicists. It is obvious that to solve this problem we need break it up into only *four* partial problems corresponding to the four faces of our solid. In two of these partial problems the exceptional face will be a cylindrical one; in the other two, one of the ends; and we see *that the partial problem of type* 3) *is no longer needed.* Moreover it will readily appear that *the partial problems of types* 1) *and* 2) *need not be modified in any essential respect.* The only change will be that the Fourier's series will proceed now according to the *sines and cosines* of multiples of φ, not merely, as before, according to the sines. Moreover, the problem will in so far lose its generality as the Bessel's functions which occur will have *integral indices.**

We may specialise our general solid S in still another way, namely by letting the radius of the inner cylindrical face become smaller and smaller and finally vanish. We shall then have but one partial problem of type 1); and the Bessel's functions involved in this problem, as well as in the partial problems of type 2), will be the *Bessel's functions with positive index* $J_{+n}(x)$ (or, in case n is a whole number, the Bessel's functions of the *first kind*). The solution of these problems will however remain otherwise unaltered.

The solution of problems of type 3) will, however, in the special case we have just considered, assume an entirely new appearance. In the first place we may notice that it is impossible to determine ν in such a way that $E_{i\nu}(ix)$ shall vanish when $x = 0$; for, whatever real value ν may have, $E_{i\nu}(ix)$ will have an infinite number of oscillations in the neighborhood of the point $x = 0$, and will accordingly have *no* assignable value when $x = 0$. Secondly, however, we may notice that it is no longer necessary that $E_{i\nu}(ix)$ should vanish when $x = 0$ but merely that it should not become infinite,† and *every* real value of ν satisfies this condition. Instead, then, of allowing ν to take on a series of distinct values to each of which corresponds a term in the development of F (see p. 147), we must now let ν take on in succession *all* positive real values from 0 to

* On the other hand, a problem practically equivalent to *type* 1) *in all its generality* is frequently treated in the theory of sound. See p. 142.

† This is of course due to the fact, that $x = 0$, or what amounts to the same thing, $r = 0$, no longer represents a real *face* of our solid, but only an *edge*.

∞, and thus the development in series of p. 147 goes over into a definite integral,

$$F(r) = \int_0^\infty a(\nu) \,.\, E_{i\nu}\left[\frac{ix\pi}{z_0} r\right] . \, d\nu \, ,$$

where the coefficient $a(\nu)$ (which, if expressed, would of course itself be a definite integral) may be determined as

$$\underset{r_1=0}{\text{limit}} \left[\frac{A_{\kappa,\mu}}{\nu_{\kappa,\mu} - \nu_{\kappa,\mu-1}}\right].$$

This will be seen to be an expression for $F(r)$ similar to Fourier's integral, except that here Bessel's functions are used instead of trigonometric, and also that the limits of integration are different.

Corresponding to this, one of the signs of summation in the solution of each of the partial problems of type 3) (*see the double series on p.* 146) *will now become an integral sign.**

One more special case of our solid S may be briefly mentioned; namely, that in which the radius of the larger curved surface of S becomes infinitely large, while the radius of the smaller cylinder retains a finite value. Here the partial problems of the types 2) and 3) will obviously suffer no essential change, (although in the problems of type 3) we have to deal with those Bessel's functions which vanish when the argument becomes infinite, of which we spoke in the note on p. 150). Here, however, in the problems of type 1), our series will have become an integral, just as was previously the case for the problems of type 3).† A special case of these integrals has already been discussed by Prof. C. Neumann; namely, that in which the radius of the outer cylinder is infinite, and at the same time the radius of the inner cylinder is zero.

It is not necessary here to speak of the other special cases of our general potential problem, in which, for instance, z_0 becomes infinite; for, although this would in some of the partial problems cause a series to degenerate into an integral, this integral would be merely a Fourier's integral.

§ 8. In the preceding sections we have learned that not merely Bessel's functions with real index, and real or pure imaginary argument, find an application in mathematical physics, but also those with pure imaginary index and

* The case just discussed may be taken as showing the application of the figure on p. 61 of my essay quoted on p. 138. I mention this because, as I stated in the closing lines of that essay, I know of no case which has been discussed where the principle illustrated by that figure finds an application.

† The remark made in the preceding note applies here also.

pure imaginary argument. It is now natural for us to ask ourselves: Have not Bessel's functions with pure imaginary index but *real* argument also a place in mathematical physics? Such functions are $H_{i\nu}(x)$ and $I_{i\nu}(x)$ which we have already briefly considered on p. 145. Applications of these functions do exist, as I shall presently show, although not to the potential problem considered up to this time.

Bessel's functions with index $i\nu$ and argument x satisfy the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 + \nu^2)\, y = 0,$$

or if we introduce the new variable $t = \log x$,

$$\frac{d^2y}{dt^2} = -(x^2 + \nu^2)\, y.$$

By means of this form of Bessel's equation the nature of the surface $y = E_{i\nu}(x)$ may be discussed in a qualitative way, as was suggested for the case of a pure imaginary argument on pp. 149–151. Here $E_{i\nu}(x)$ denotes, again, that Bessel's function which vanishes when $x = x_1$; ν and x are taken as the horizontal coordinates, as before; and y as the vertical coordinate. The trace of this surface on the plane $y = 0$ will readily be found to have the general form shown in Fig. II, the portions of the surface included between the curves of this figure being, of course, alternately ridges and valleys.

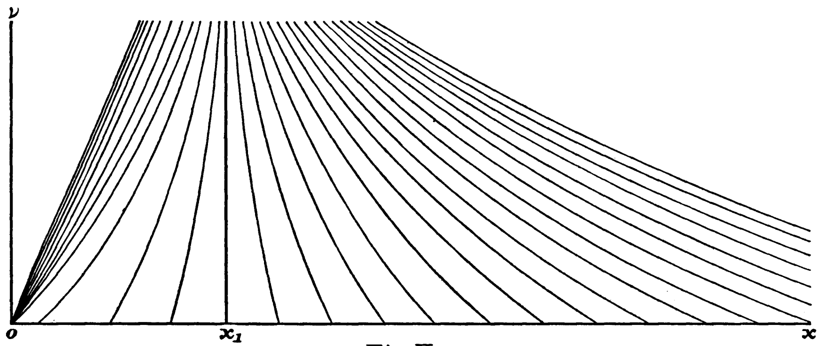

Fig. II.

Before going farther we must consider for a moment Bessel's functions with real index and argument. Let $E_n(x)$ be such a function which vanishes when x has the real value x_1. By means of Bessel's equation in the form

$$\frac{d^2y}{dt^2} = -(x^2 - n^2)\, y,$$

we can, as before, easily get an idea of the surface $y = E_n(x)$ (x, n, and y being the coordinates); and we see that its trace on the plane $y = 0$ will be qualitatively represented by Fig. III. In connection with this figure we may note that the greater the value of x_1 the greater will be the number of curves between the lines $x = 0$ and $x = x_1$.

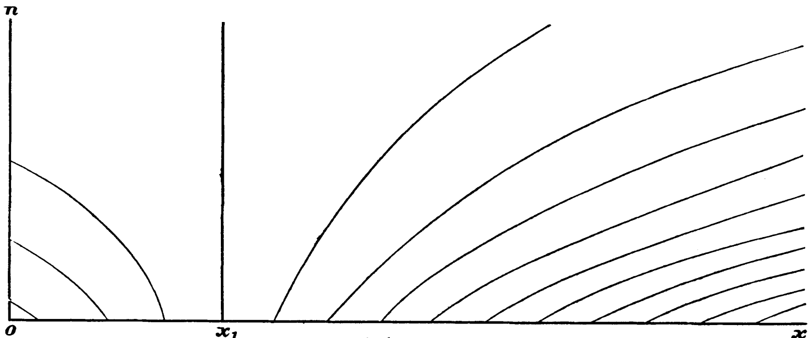

Fig. III.

By means of Figs. II and III we now see, at once, the truth of the following proposition, in which $E_n(x)$ is supposed to be a Bessel's function which vanishes when x has the real value x_1:—

The transcendental equation $E_n(x_2) = 0$, *where* x_2 *is a real constant has:* (1) *an infinite number of pure imaginary roots, and,* (2) *a finite number of real roots, which number, however, will be the smaller the shorter the interval between* x_1 *and* x_2. *If this interval is sufficiently short there will even be no real roots.**

Let us now call the largest of these real roots n_1, the next smaller n_2, etc., and the smallest of them n_λ; and then beginning with the smallest of the pure imaginary roots, call it $n_{\lambda+1} = i\nu_{\lambda+1}$, call the next larger $n_{\lambda+2} = i\nu_{\lambda+2}$, etc. We have, then, as we again see from our figures, the following *theorem of oscillation:*—

The equation $E_{n_\mu}(x) = 0$ *has* $\mu - 1$ *real roots between the values* $x = x_1$ *and* $x = x_2$, *whether* n_μ *is one of the real or one of the pure imaginary roots above mentioned.*

A simple problem which introduces the Bessel's functions we have just been considering is the following:—

Given a plane lamina bounded by two arcs of concentric circles and two radii of these circles, to determine a function $u(x, y)$, *which* (1) *within the lamina satisfies the differential equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$, *and is finite, sin-*

* Here again, as on p. 152, it can easily be proved that no complex roots exist.

gle valued, and continuous, together with its first derivatives, and (2) *assumes on the boundary of the lamina arbitrarily assigned values.*

This problem is briefly discussed on pp. 330–331 of Dr. Pockels's recently published book,* to which I must refer the reader for an account of the physical questions in connection with which this problem comes up. The first step towards the solution of this problem consists in breaking it up into four partial problems, in each of which the function u is required to assume arbitrarily assigned values on one side only of the lamina (the *exceptional* side), while on the other three sides it is required that the function u should vanish. We thus get two types of problem, according as the exceptional side is 1) an arc of a circle, or 2) a part of a radius. These two types of problem require a separate treatment; and in fact, each problem will have a somewhat different character according as the constant k, which occurs in the partial differential equation satisfied by u, has a real or a pure imaginary value.

In any case, however, we must first transform our differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

into polar coordinates (r, φ), and then attempt to satisfy it by means of Lamé's products

$$u = R \,.\, \Phi,$$

where the factors R and Φ are functions, respectively, of r and φ alone. It is easy to see that this can be done when, and only when, R and Φ have the following form:

$$R = A_1 J_n(kr) + B_1 J_{-n}(kr),$$
$$\Phi = A_2 \sin(n\varphi) + B_2 \cos(n\varphi).$$

In each of the partial problems we have then to choose out those Lamé's products which satisfy all the boundary conditions except the one on the exceptional side, and from these Lamé's products to build up the required solution in the form of a series. We will suppose that the sides of the lamina are represented in polar coordinates by the equations

$$\varphi = 0, \quad \varphi = \varphi_0, \quad r = r_1, \quad r = r_2.$$

As a partial problem of type 1) we will take the one in which $r = r_2$ is the exceptional side. In order that the Lamé's products should vanish when $\varphi = 0$, $\varphi = \varphi_0$, and $r = r_1$, we must let

$$B_2 = 0, \quad n = \frac{\nu\pi}{\varphi_0}, \quad \frac{A_1}{B_1} = -\frac{J_{-n}(kr_1)}{J_n(kr_1)},$$

where ν is a real integer, which we may obviously take as positive.

* Über die Differentialgleichung $\Delta u + k^2 u = 0$. Friedrich Pockels, mit einem Vorwort von F. Klein. Teubner, Leipzig, 1891.

Calling $E_n(kr)$ the Bessel's functions thus determined (which vanish when $r = r_1$), we must next form the series

$$u = \sum_{1}^{\infty}{}_{\nu} A_{\nu} \sin\left[\frac{\nu\pi}{\varphi_0}\varphi\right] . E_{\frac{\nu\pi}{\phi_0}}(kr) ,$$

which will be the required solution of our problem, if the coefficients A_ν are so determined that when $r = r_2$ the series shall reduce to a development of the arbitrarily assigned function $F(\varphi)$. When $r = r_2$ we shall, however, merely have a Fourier's series, whose coefficients can be determined by the ordinary formula.

In case k has a pure imaginary value, this problem is very similar to type 2) of our potential problem which was discussed in § 3. If, however, k is real, certain complications may arise, owing to the fact that some of the terms in the development of u may vanish, not only when $r = r_1$, but also when $r = r_2$.*

This brief treatment of the problem of type 1) is identical with that given by Dr. Pockels. In the problem of type 2), however, while the same author mentions the fact that we shall have to deal with Bessel's functions with pure imaginary index,† there still seems to be a not unimportant oversight in the very brief treatment of the problem there given.

As a problem of type 2) we will take the partial problem in which the side $\varphi = \varphi_0$ is the exceptional one. In order, then, that the Lamé's products should vanish when $r = r_1$, $r = r_2$, and $\varphi = 0$, we must let

$$B_2 = 0, \quad \frac{A_1}{B_1} = -\frac{J_{-n}(kr_1)}{J_n(kr_1)} ,$$

and take n as a root of the transcendental equation

$$E_n(kr_2) = 0 ,$$

where $E_n(kr)$ is a Bessel's function which vanishes when $r = r_1$.

In case k has a pure imaginary value, this last written equation has an infinite number of pure imaginary roots, and no others; and the solution of the problem will be very similar to that of type 3) of our potential problem discussed in § 5, and it will accordingly not be necessary for us to consider it here.

* Concerning this case in which the value of k is an exceptional (ausgezeichnete) one for the lamina in question I must again refer to Dr. Pockels's book.

† This is the only place in which I have seen it even suggested that Bessel's functions with pure imaginary index can occur in physical investigations, and even here the nature of these functions is not even hinted at, nor are any formulæ given concerning them.

If, however, k is real, as we will henceforth suppose it to be, the equation $E_n(kr_2) = 0$ will have, as we have seen, not merely an infinite number of pure imaginary roots, but also, in general, a finite number of real roots. The presence of these real roots seems to have escaped Dr. Pockels's notice. Owing to their presence, however, it may, and for certain values of k will, happen that some of the Lamé's products we have just picked out will vanish on all four sides of our lamina (i. e. not merely when $\varphi = 0$, but also when $\varphi = \varphi_0$).* Such special cases, however, need not here concern us.

Finally we build up from the Lamé's products which we have thus selected a series of the form,

$$u = \sum_1^{\lambda}{}^{\mu} A_\mu \,.\, \sin\,(n_\mu \,.\, \varphi) \,.\, E_{n_\mu}(kr) + \sum_{\lambda+1}^{\infty}{}^{\mu} A_\mu \sinh\,(\nu_\mu \,.\, \varphi) \,.\, E_{i\nu_\mu}(kr)\,,$$

in which the quantities $n_\mu = i\nu_\mu$ are the roots of the transcendental equation just considered arranged as was indicated on p. 156. The determination of the coefficients in this development may be performed by means of the formula given on p. 147, as the proof of this formula did not depend upon the reality of either argument or index.

Two special cases remain to be mentioned:

(*a*) We may let $r_2 = \infty$ while r_1 remains finite. Here any value of n, whether real or pure imaginary, will cause an infinite number of oscillations of our Bessel's functions in the neighborhood of the point $r = \infty$. Accordingly *any* value of n will satisfy the condition of not making $E_n(kr)$ infinite when $r = \infty$, and this (as on p. 153) is all we can now require. We see, then, that each part of the series we had in the general case degenerates into an integral, so that our solution now takes on the form

$$u = \int_0^\infty a(n) \,.\, \sin\,(n\varphi) \,.\, E_n(kr) \,.\, dn + \int_0^\infty b\,(\nu) \,.\, \sinh\,(\nu\varphi) \,.\, E_{i\nu}(kr) \,.\, d\nu\,.$$

(*b*) If $r_2 = 0$ while r_1 retains a finite value, we get a special case of peculiar interest. Here our transcendental equation retains, as in the general case, a finite number of real roots and the corresponding functions $E_{n_1}(kr)$, $E_{n_2}(kr)$, $\ldots$, $E_{n_\lambda}(kr)$ are now obviously identical with the ordinary Bessel's functions $J_{n_1}(kr)$, $J_{n_2}(kr)$; $\ldots$, $J_{n_\lambda}(kr)$.† The infinite number of pure imaginary roots

* Dr. Pockels expressly states that this cannot occur.

† The indices n_1, n_2, $\ldots$, n_λ may now be determined, somewhat more simply than before, as the real roots of the equation $J_n(kr_1) = 0$.

of our equation have, however, ceased to exist, since when $r = 0$ Bessel's functions with pure imaginary index have no assignable value. These roots are, however (just as in the similar cases we have already considered) now replaced by *all* pure imaginary values of n. This part of our series will therefore degenerate into an integral, and our solution will have the form

$$u = \sum_{1}^{\lambda} {}^{\mu} A_\mu \sin (n_\mu . \varphi) . J_{n_\mu}(kr) + \int_0^\infty b(\nu) . \sinh (\nu\varphi) . E_{i\nu}(kr)\, d\nu .$$

This remarkable form of a degenerate series in which the first few terms retain the form of a series, while the rest degenerate into an integral is, I think, new.*

* Similar cases of degeneration will be found to occur in certain potential problems somewhat more complicated than the one considered in this paper; for instance, in the potential problem concerning an infinite solid bounded by an unparted hyperboloid of revolution and a prolate spheroid confocal with it.

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